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LETTER TO THE EDITOR

About polynomials related to multiphotonic bremsstrahlung effects

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Abstract. The symmetric convolution of two Hermite functions appears as transition amplitude in multiphotonic Bremsstrahlung effect analysed by quantum oscillator models. We identify these transition amplitudes in terms of Laguerre polynomials and we give new recurrence relations and a differential equation for these physical quantities.

Polynomials obtained from the symmetric convolution of two Hermite functions have recently been given explicitly [1]. These polynomials appear in multiphotonic Bremsstrahlung effects using harmonic oscillator models with sudden displacement of the equilibrium point x_0. The aim of this letter is to identify these polynomials as the Laguerre polynomial which explain immediately the Bessel-type asymptotic behaviour. We give also a more direct derivation of these polynomials, a differential equation, and a pure recurrence relation for the transition amplitude.

The transition amplitude between two states n and $n+q$ is given by [1, 2]:

$$C_{n,n+q}(x_0)=\int_{-\infty}^{+\infty}\psi_n(x)\psi_{n+q}(x-x_0)\,\mathrm{d}x \tag{1}$$

where $\psi_n(x)$ are the normalized wavefunctions of the one-dimensional harmonic oscillator:

$$\psi_n(x)=(2^n n!\sqrt{\pi})^{-1/2}\exp(-x^2/2)H_n(x) \tag{2}$$

and $H_n(x)$ is the Hermite polynomial of degree n normalized in the usual way:

$$\int_{-\infty}^{+\infty}[H_n(x)]^2\exp(-x^2)\,\mathrm{d}x=\pi^{1/2}2^n n!.$$

The result given in [2] for $n=1$ is extended to any n in [1] in the following form:

$$C_{n,n+q}(x_0)=\left(\frac{n!(n+q)!}{2^q}\right)^{1/2}\exp(-x_0^2/4)(-x_0)^q\sum_{k=0}^{n}\frac{(-x_0^2/2)^k}{k!(n-k)!(q+k)!}. \tag{3}$$

Using the definition of the Laguerre polynomials [3]

$$L_n^{(q)}(s)=\sum_{k=0}^{n}(-1)^k\frac{(n+q)!}{(n-k)!(q+k)!}\frac{s^k}{k!} \tag{4}$$

we can write the transition amplitude (3) as

$$C_{n,n+q}(x_0)=\left(\frac{n!}{2^q(n+q)!}\right)^{1/2}\exp(-x_0^2/4)(-x_0)^q L_n^{(q)}\left(\frac{x_0^2}{2}\right) \tag{5}$$

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from which the asymptotic behaviour is immediately obtained from the well known limit relation of the Laguerre polynomials [3]:

$$\lim_{n\to\infty}\left[n^{-q}L_n^{(q)}\left(\frac{s}{n}\right)\right]=s^{-q/2}J_q(2\sqrt{s}) \tag{6}$$

where $J_q(x)$ is the usual Bessel function. With

$$\frac{n!}{(n+q)!}\sim n^{-q/2} \tag{7}$$

and using the author notation [1] $x_0=y/\sqrt{2n}$, equations (5) and (6) with $s=y^2/4$ give:

$$C_{n,n+q}\left(\frac{y}{\sqrt{2n}}\right)\to(-1)^qJ_q(y) \tag{8}$$

Relation (5) can be proved directly in the following way. Starting from (1)

$$\begin{aligned}C_{n,n+q}(x_0)&=\frac{1}{\sqrt{\pi}}[2^{2n+q}n!(n+q)!]^{-1/2}\int_{-\infty}^{+\infty}\exp\left(-\frac{x^2}{2}-\frac{(x-x_0)^2}{2}\right)H_n(x)H_{n+q}(x-x_0)\,\mathrm{d}x\\&=\frac{1}{\sqrt{\pi}}[2^{2n+q}n!(n+q)!]^{-1/2}\exp(-x_0^2/4)\int_{-\infty}^{+\infty}\mathrm{d}s\exp(-s^2)\\&\quad\times H_n\left(s+\frac{x_0}{2}\right)H_{n+q}\left(s-\frac{x_0}{2}\right)\end{aligned} \tag{9}$$

with $s=x-x_0/2$.

The Taylor–Maclaurin development of $H_n(s\pm x_0/2)$ in the integral gives:

$$\begin{aligned}&=\frac{1}{\sqrt{\pi}}[2^{2n+q}n!(n+q)!]^{-1/2}\exp(-x_0^2/4)\int_{-\infty}^{+\infty}\mathrm{d}s\exp(-s^2)\\&\quad\times\left[\sum_{k=0}^{n}\frac{H_n^{(k)}(s)}{k!}\left(\frac{x_0}{2}\right)^k\right]\left[\sum_{r=0}^{n+q}\frac{H_{n+q}^{(r)}(s)}{r!}\left(\frac{x_0}{2}\right)^r(-1)^r\right].\end{aligned} \tag{10}$$

Now the Appel property of the Hermite polynomials:

$$\begin{aligned}&H_n'=2nH_{n-1}\\&H_n^{(k)}=2^k\frac{n!}{(n-k)!}H_{n-k}\end{aligned} \tag{11}$$

and the orthogonality property of the Hermite polynomials reduces the double sum to the single sum (3) $(r=q+k)$.

From the three-term recurrence relation of the Laguerre polynomials:

$$nL_n^{(q)}(s)=(2n+q-1-s)L_{n-1}^{(q)}(s)-(n+q-1)L_{n-2}^{(q)}(s) \tag{12}$$

we deduce the pure recurrence relation between the transition amplitude $C_{n,n+q}(x_0)$:

$$\begin{aligned}n\left[\frac{(n+q)(n+q-1)}{n(n-1)}\right]^{1/2}C_{n,n+q}(x_0)&=\left(2n+q-1-\frac{x_0^2}{2}\right)\left[\frac{n+q-1}{n-1}\right]^{1/2}C_{n-1,n+q-1}(x_0)\\&\quad-(n+q-1)C_{n-2,n+q-2}.\end{aligned} \tag{13}$$

Now from the so-called 'structure relation' [4]

$$-s\frac{\mathrm{d}}{\mathrm{d}s}L_{n+1}^{(q)}(s)=(n+q+1)L_n^{(q)}(s)-(n+1)L_{n+1}^{(q)}(s) \tag{14}$$

we obtain

$$\frac{x_0}{2}\left(\frac{n+q+1}{n+1}\right)^{1/2} C'_{n+1,n+q+1}(x_0)$$
$$= \left[[(n+q+1)(n+1)]^{1/2} + \left(\frac{x_0^2}{4}-\frac{q}{2}\right)\left(\frac{n+q+1}{n+1}\right)^{1/2}\right] C_{n+1,n+q+1}(x_0)$$
$$-(n+q+1)C_{n,n+q}(x_0) \tag{15}$$

where $\mathrm{d}/\mathrm{d}x_0$ is replaced by a 'prime'.

A differential equation can also be produced. From:

$$s\frac{\mathrm{d}^2}{\mathrm{d}s^2}[L_n^{(q)}(s)] - (s-q-1)\frac{\mathrm{d}}{\mathrm{d}s}L_n^{(q)}(s) + nL_n^{(q)}(s) = 0 \tag{16}$$

and replacing in (5), we get:

$$x_0^2 C''_{n,n+q}(x_0) + x_0 C'_{n,n+q}(x_0) - \left[\frac{x_0^4}{4} - (q+2n+1)x_0^2 + q^2\right] C_{n,n+q}(x_0) = 0. \tag{17}$$

The formulae (5) and (9) give a representation of the Laguerre polynomial as convolution of two Hermite functions which appear as a particular case of an integral given in [5] p 503 equation (10):

$$L_n^{(q)}(s) = N_{nq}(s)\int_{-\infty}^{+\infty} \exp(-x^2) H_n(x+\sqrt{s/2}) H_{n+q}(x-\sqrt{s/2})\,\mathrm{d}x \tag{18}$$

with

$$N_{nq}(s) = [2^{n+q}\sqrt{\pi}\,n!(-\sqrt{s/2})^q]^{-1}$$

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